Overview of Kalman Filter Theory and Navigation Applications
Day 1
Supplement

Michael L. Carroll

Feb 24, 2004

©2003 by Michael L. Carroll

Day 1: The Basic Kalman Equations, Part I
Segments

- Ionospheric Probe (Gelb)

- Baro-Inertial Vertical Channel

- Simulink Example Geogrphic Position Rate Equations

Ionospheric Probe

- Balloon Launched Ionosphereic Probe
- Example originally due to Gelb
- Grewal and Andrews have also presented it
- Illustrates single-axis, short-term inertial error dynamics

Ionosphereic Probe: Continuous Model

(1) $$\delta p(t) \approx \delta p(0) + \delta v(0)t + \delta a(0)\frac{t^2}{2}$$

where δp =position error, δv =velocity error, δa =acceleration error.

Assume $\delta \dot{a} = 0$.

State model: $x_1(t) = \delta p(t)$, $x_2(t) = \dot{x}_1(t)$, $x_3(t) = \dot{x}_2(t)$

Ionospheric Probe: Model $\dot{x} = Fx$

State Dynamics Model (with companion matrix):

(2)

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

or

$$\begin{bmatrix} \delta\dot{p} \\ \delta\dot{v} \\ \delta\dot{a} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \delta p \\ \delta v \\ \delta a \end{bmatrix}$$

State Transistion Matrix $\Phi(t)$

As matrix exponential $\Phi(t) = \exp Ft = e^{Ft}$

(3) $$\Phi(t) = e^{Ft} = I + Ft + \frac{F^2t^2}{2!} + \frac{F^3t^3}{3!} + \cdots = \begin{bmatrix} 1 & t & \frac{t^2}{2} \\ 0 & 1 & t \\ 0 & 0 & 1 \end{bmatrix}$$

because $F^n = 0$ for all $n \geq 3$.

Note also that here $\Delta t = t - 0 = t$ in this example.

Ionospheric Probe: Measurement Model

- Radio-indicated position and INS indicated position.
- Scalar measurement formed by differencing the two indicated positions

$$\begin{aligned} z &= p_{\text{radio}} - p_{\text{INS}} \\ &= (p_{\text{true}} + \delta p_{\text{radio}}) - p_{\text{true}} + \delta p_{\text{INS}}) \\ &= -\delta p + \delta p_{\text{radio}} \end{aligned}$$

Thus, $H = \begin{bmatrix} -1 & 0 & 0 \end{bmatrix}$ and measurement noise $v = \delta p_{\text{radio}}$

Exercise: Solve the discrete Kalman equations for K_k, $P_k(+)$ and $\widehat{x}_k(+)$ using

$t_0 = 0.0$, $t_1 = 0.5$, $t_2 = 1.0$

The equations to solve are:

$$K_k = P_k(-)H_k^{\mathsf{T}} \left[H_k P_k(-) H_k^{\mathsf{T}} + R_k\right]^{-1}$$

$$P_k(+) = \left[I - K_k H_k\right] P_k(-)$$

$$\widehat{x}_k(+) = \widehat{x}_k(-) + K_k \left[z_k - H_k \widehat{x}_k(-)\right]$$

Assume $\widehat{x}_0(+) = 0$, $P_0(+) = \begin{bmatrix} p_{11}(0) & 0 & 0 \\ 0 & p_{22}(0) & 0 \\ 0 & 0 & p_{33}(0) \end{bmatrix}$ with $p_{11}(0) = (1\text{nm})^2$, $p_{22}(0) = (7.3\text{kts})^2$, $p_{33}(0) \neq 0$. Also, assume measurement noise standard deviation is $\sigma_{\text{radio}} = 30\text{ft}$

Ionospheric Probe: Solution Approach

- First extrapolate $\widehat{x}_0(+)$ to time $t_1 = 30$ sec using $\Phi(30)$ to get $\widehat{x}_1(-)$

- Extrapolate $P_0(+)$ to get $P_1(-)$ using $\Phi(30)$ and $\Phi^{\mathsf{T}}(30)$. There is no process noise in this problem.

- Compute Kalman gain at $t_1 = 30$

- Update the state estimate and the error covariance.

Baro-Inertial Vertical Channel

- Plumb-Bob Gravity and Gravity Modeling
- Vertical Channel

Plumb-Bob Gravity

- Gravitational acceleration as would be sensed by a plumb-bob: true gravity plus centripetal force

- Changes with location and altitude

- Very closely aligned with geodetic vertical
  - geodetic vertical: normal to ellipsoid

$$g' = g - \Omega_e \times (\Omega_e \times r)$$

Plumb-Bob Gravity

- Horizontal components ≈ 0 at earth's surface in local level coordinate frame
- Altitude dependent term can be added

Gravity Model

$$
\begin{aligned}
(4) \qquad g' &= G_1\left(1 - 2\frac{h}{R_e} + 2e\sin^2\phi\right) \\
&\quad + \frac{3}{2}J_2 G_1(1 - 3\sin^2\phi) \\
&\quad - R_e\,\|\Omega_e\|^2\,(1 - e\sin^2\phi\frac{h}{R_e})\cos^2\phi
\end{aligned}
$$

where

$g' =$ plumb-bob gravity magnitude

$G_1 =$ mean gravity magnitude at earth's surface on equator

$h =$ altitude

$R_e =$ mean earth equatorial radius

$e =$ earth's ellipticity 1/298

$\phi =$ geodetic latitude

$J_2 =$ empirical constant $= 0.00108$

$\Omega_e =$ earth rate vector

Baro-Inertial Vertical Channel

- Inertial errors and gravity model inaccuracies conspire to make the vertical channel unstable
  - Integrated outputs of Z accelerometer yield altitude errors
  - But the gravity model is a function of altitude!
  - Vicious circle: erroneous gravity enters into integration of vertical acceleration

Baro-Inertial Vertical Channel

- Stabilize loop by using difference between barometric altitude and INS-indicated altitude
- Baro altitude has long-term stability
- INS altitude has wide bandwidth, i.e., responds well to sudden changes in altitude
- Difference between baro and inertial altitude is used as a feedback signal

Baro−Inertial Vertical Channel
(simple model with constant inputs)